Non-Null Intersection Curves of Timelike Surfaces in Lorentz-Minkowski 3-Space

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Abstract— **In this paper, we investigate the curvature κ and the torsion τ of a intersection curve of two timelike surfaces in Lorentz-Minkowski 3-space.**

Index Terms—**curvature, intersection curve, torsion.**

I. INTRODUCTION

The surface-surface intersection (SSI), is a fundamental problem in computational geometry and geometric modeling of complex shapes. In general, what it is wanted in such problems is to determine the intersection curve between two given surfaces. To compute the intersection curve with precision and efficiency, it is necessary to obtain the geometric properties (such as curvature and torsion) of the intersection curves [1].

Ye and Maekawa[4] provides t, n, b, κ, τ and algorithms for the evaluation of higher order derivatives for transversal as well as tangential intersections for all types intersection problems. From the motivation Allesio and Guadalupe [1] studied intersection curve of two parametric spacelike surfaces in Lorentz-Minkowski 3-space L^3.

In the present paper we investigate the curvature κ and the torsion τ of a intersection curve of two timelike surfaces in Lorentz-Minkowski 3-space.

II. PRELIMINARIES

Lorentz-Minkowski space is the metric space $L^3=(R^3,<,>)$, where $u=(u_1,u_2,u_3), v=(v_1,v_2,v_3) \in R^3$, the metric <,> is given by

$$<u,v>=u_1v_1+u_2v_2-u_3v_3. \tag{1}$$

The metric <,> is called the Lorentzian metric. Recall that a vector $v \neq 0$ in L^3 can be a spacelike, a timelike or a null (lightlike), if respectively holds $<v,v>> 0$, $<v,v>< 0$ or $<v,v>=0$. In particular, the vector $v=0$ is a spacelike. If $v=(v_1,v_2,v_3)$ is in L^3 we define the norm of v by

$$\|v\| = \sqrt{|\langle v,v\rangle|} = \sqrt{|v_1^2+v_2^2-v_3^2|} \tag{2}$$

The vectors u and v in L^3 are said to be orthogonal if $<u,v>=0$. A vector u in L^3 which satisfies $<u,u>=\pm 1$ is called a unit vector[2].

We also recall that the vector product [1] of u and v (in that order) is the unique vector $u \times v \in L^3$ defined by

$$u \times v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{3}$$

where $\{e_1,e_2,e_3\}$ is the canonical basis of L^3 and $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$. We can easily check that the triple scalar product of the three vectors u,v,w is given by

$$\langle w, u\times v\rangle = \det(w,u,v) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{4}$$

where $w=(w_1,w_2,w_3)$. Recall that the vector product is not associative, and that moreover we have the following properties

$$\langle u\times v, x\times y\rangle = \begin{vmatrix} \langle u,y\rangle & \langle v,y\rangle \\ \langle u,x\rangle & \langle v,x\rangle \end{vmatrix} \tag{5}$$

where $u,v,x,y \in L^3$. If u and v are timelike vectors of L^3 then the vector $u\times v$ can be a spacelike, a timelike or a null vector of L^3. We also recall that an arbitrary curve $\alpha=\alpha(s)$ can locally be a spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. [3] A non null curve $\alpha(s)$ is said to be parametrized by pseudo arclenght parameter s, if hold $<\alpha'(s),\alpha'(s)>=\pm 1$. In this case, the curve α said to be of unit speed.

We suppose that α is a timelike curve. Then $\alpha''(s) \neq 0$ is a spacelike vector independent with $\alpha'(s)$. We define the curvature of α at s as $\kappa(s)=\|\alpha''(s)\|$. The normal vector $n(s)$ is defined by

$$n(s) = \frac{t'(s)}{\kappa(s)}.$$

Moreover $\kappa(s)=\langle t'(s), n(s)\rangle$. We call binormal vector $b(s)$ as

$$b(s) = t(s)\times n(s).$$

The vector $b(s)$ is unitary and spacelike. For each s, $\{t,n,b\}$ is an orthonormal base of L^3 which called the Frenet trihedron of α. We define the torsion of α at s as

$$\tau(s) = \langle n'(s), b(s)\rangle.$$

By differentiation each one of the vector functions of the Frenet trihedron and putting in relation with the same Frenet base, we obtain the Frenet equations, namely[2]

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$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{6}$$

Let $\alpha = \alpha(s)$ be a spacelike curve parametrized by arclenght s. Therefore $\alpha'(s)$ is a spacelike unit vector, ie, $\|\alpha'(s)\|^2 = 1$. Then

$$< \alpha''(s), \alpha'(s) >= 0.$$

Then depending of the vector $\alpha''(s)$ we consider following three cases[2]:

Case 1. $< \alpha''(s), \alpha''(s) >> 0$. Again we write $\kappa(s) = \|\alpha''(s)\|$, $n(s) = \frac{t'(s)}{\kappa(s)}$ and $b(s) = t(s) \times n(s)$. The vectors n and b are called the normal vector and the binormal vector respectively. The curvature of α defined by κ. The Frenet equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{7}$$

The torsion of α defined by $\tau = -< n', b >$.

Case 2 . $< \alpha''(s), \alpha''(s) >< 0$. The normal vector is $n = t'/\kappa$, where $\kappa(s) = \sqrt{-< \alpha''(s), \alpha''(s) >}$ is the curvature of α. The binormal vector is $b = t \times n$, which is a spacelike vector. Now, the Frenet equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{8}$$

The torsion of α is $\tau =< n', b >$.

Case 3. $< \alpha''(s), \alpha''(s) >= 0$. We define the normal vector as $n(s) = t'(s)$, which is independent linear with $t(s)$. Let b be the unique lightlike vector such that $< n, b >= 1$ and orthogonal to $t(s)$. The vector $b(s)$ is called the binormal vector of α at s. The Frenet equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & \tau \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{9}$$

The function τ is called the torsion of α. There is not a definition of the curvature of α.

Now let us evaluate the third derivative $\alpha'''(s)$. By differentiating equation α'', we obtain

$$\alpha''' = \kappa' n + \kappa n'$$

for timelike curves and in the first two cases for spacelike curves and

$$\alpha''' = \tau n$$

for the third case for spacelike curves, where we can replace n' by the second equation of the Frenet formulas. Then we have

$$\alpha''' = \kappa^2 t + \kappa' n + \kappa\tau b \tag{10}$$

for timelike curves and

$$\text{case1: } \alpha''' = -\kappa^2 t + \kappa' n + \kappa\tau b \tag{11}$$

$$\text{case2: } \alpha''' = \kappa^2 t + \kappa' n + \kappa\tau b \tag{12}$$

$$\text{case3: } \alpha''' = \tau n \tag{13}$$

for spacelike curves.

The torsion can be obtained from Eq.(10), Eq.(11), Eq.(12) and Eq.(13) as

$$\tau = \frac{<\alpha''', b>}{\kappa} \tag{14}$$

for timelike curves and

$$\text{case1: } \tau = -\frac{\langle \alpha''', b \rangle}{\kappa} \tag{15}$$

$$\text{case2: } \tau = \frac{\langle \alpha''', b \rangle}{\kappa} \tag{16}$$

$$\text{case3: } \tau =< \alpha''', b > \tag{17}$$

for spacelike curves[1].

Recall that an arbitrary plane in L^3 is timelike if the induced metric is Lorentzian. We also recall that any arbitrary regular surface $X = X(u, v)$ is called timelike surface if the tangent plane at any point is timelike[3].

The surface normal is perpendicular to the tangent plane and hence at any point the unit normal vector is given by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

therefore N is a timelike unit vector of L^3[2].

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III. NON-NULL INTERSECTION CURVES

In this section we will give a characterization for intersection curve of two timelike surfaces and compute its the curvature κ and torsion τ of this curve.

Let $X^A = X^A(u, v)$ and $X^B = X^B(u, v)$ be the two timelike parametric surfaces. Let $\alpha = \alpha(s)$ be the intersection curve of both surfaces X^A and X^B. Then the tangent vector of the intersection curve lies on the timelike tangent planes of both surfaces. Therefore it can be as the cross product of the unit surface normal of the surfaces at $p = \alpha(s)$

$$t = \frac{N^A \times N^B}{\|N^A \times N^B\|}$$

where N^A and N^B are the spacelike unit normal vectors to timelike surfaces X^A and X^B respectively. On the other hand, since t lies timelike planes of both surfaces X^A and X^B, it can be spacelike, timelike or null. Now let assume that $\langle N^A, N^B \rangle = \lambda$. Then from Eq.(5) we have

$$\langle N^A \times N^B, N^A \times N^B \rangle = \lambda^2 - 1.$$

Thus we have following characterization:

Proposition 3.1. Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be the two timelike parametric surfaces in Lorentz-Minkowski 3-space L^3 and N^A and N^B are the spacelike unit normal vectors to timelike surfaces X^A and X^B, respectively. Let $\alpha = \alpha(s)$ be the intersection curve of both surfaces X^A and X^B. Then

1. α(s) is spacelike curve if $\lambda \in (-1,1)$
2. α(s) is timelike curve if $\lambda \in R - [-1,1]$
3. α(s) is null curve if $\lambda = \pm 1$

where $\langle N^A, N^B \rangle = \lambda$.

The curvature vector α'' of the intersection curve at p, being perpendicular to t, must lie in the normal plane spanned by N^A and N^B. Thus we can express it as

$$\alpha'' = aN^A + bN^B \tag{18}$$

where a and b are the coefficients that we need to determine. We know that normal curvature at p in the direction t is the projection of the curvature vector $\alpha'' = \kappa n$ onto the spacelike unit surface normal N at p, therefore by projecting Eq.(18) onto the spacelike normals of both surfaces we have

$$\begin{aligned} k_n^A &= a + b\lambda \\ k_n^B &= \alpha\lambda + b \end{aligned} \tag{19}$$

Where $\langle N^A, N^B \rangle = \lambda$. Then we have the following result:

Proposition 3.2. Suppose that $\alpha = \alpha(s)$ be the non-null intersection curve of two timelike surfaces X^A and X^B and α'' spacelike or timelike vector for spacelike case. Then the curvature κ of the curve α is given by

$$\kappa^2 = \frac{\left|(k_n^A)^2 + (k_n^B)^2 - 2\lambda k_n^A k_n^B\right|}{1-\lambda^2}. \tag{20}$$

Proof: Solving the coefficients a and b from linear systems Eq.(19), we have

$$a = \left(\frac{k_n^A - \lambda k_n^B}{1-\lambda^2}\right),$$
$$b = \left(\frac{k_n^B - \lambda k_n^A}{1-\lambda^2}\right).$$

Substituting these results in Eq.(18) we have

$$\alpha'' = \left(\left(\frac{k_n^A - \lambda k_n^B}{1-\lambda^2}\right)\right) N^A + \left(\left(\frac{k_n^B - \lambda k_n^A}{1-\lambda^2}\right)\right) N^B. \tag{21}$$

Then the curvature of the non-null intersection curve α can be calculated using Eq.(21) and definition of curvature as follows.

$$\kappa^2 = \frac{\left|(k_n^A)^2 + (k_n^B)^2 - 2\lambda k_n^A k_n^B\right|}{1-\lambda^2}.$$

Since the unit spacelike unit normal vectors N^A and N^B lie on the normal plane, the term $\kappa' n + \kappa\tau b$ in Eq.(10), Eq.(11) and Eq.(12) and the term τn in Eq.(13) can be replaced by $cN^A + dN^B$. Thus

$$\alpha''' = ft + cN^A + dN^B \tag{22}$$

for timelike intersection curves and

$$\text{case1}: \alpha''' = gt + cN^A + dN^B \tag{23}$$

$$\text{case2}: \alpha''' = ht + cN^A + dN^B \tag{24}$$

$$\text{case3}: \alpha''' = cN^A + dN^B \tag{25}$$

for spacelike intersection curves where f, g, h are arbitrary real valued functions.

Now, if we project α''' onto the spacelike unit surface normal N at $\alpha(s)$ and denote by p_n, we obtain

$$\begin{aligned} p_n^A &= c + d\lambda, \\ p_n^B &= c\lambda + d. \end{aligned} \tag{26}$$

Thus we have the following:

Proposition 3.3. Suppose that $\alpha = \alpha(s)$ be the non-null intersection curve of two timelike surfaces X^A and X^B. Then the torsion τ of the curve α is given by

$$\tau = \frac{(p_n^A - \lambda p_n^B) < N^A, b > + (p_n^B - \lambda p_n^A) < N^B, b >}{\kappa(1-\lambda^2)} \tag{27}$$

if α is timelike curve, and

$$\text{case1}: \tau = \frac{-(p_n^A - \lambda p_n^B) < N^A, b > - (p_n^B - \lambda p_n^A) < N^B, b >}{\kappa(1-\lambda^2)} \tag{28}$$

$$\text{case2}: \tau = \frac{(p_n^A - \lambda p_n^B) < N^A, b > + (p_n^B - \lambda p_n^A) < N^B, b >}{\kappa(1-\lambda^2)} \tag{29}$$

$$\text{case3}: \tau = \frac{(p_n^A - \lambda p_n^B) < N^A, b > + (p_n^B - \lambda p_n^A) < N^B, b >}{(1-\lambda^2)} \tag{30}$$

if α is spacelike curve where the b is binormal vector.

Proof: Solving the coefficients and from linear systems Eq.(26), we have

$$c = \left(\frac{p_n^A - \lambda p_n^B}{1-\lambda^2}\right),$$
$$d = \left(\frac{p_n^B - \lambda p_n^A}{1-\lambda^2}\right)$$

and substituting in Eq.(22), Eq.(23), Eq.(24), and Eq.(25), and from Eq.(14), Eq.(15), Eq.(16), and Eq.(17) we have the results.

IV. TIMELIKE INTERSECTION CURVES

Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be the two timelike parametric surfaces in Lorentz-Minkowski 3-space L^3 and N^A and N^B are the spacelike unit normal vectors to timelike surfaces X^A and X^B, respectively. Let $\alpha = \alpha(s)$ be the intersection curve of both surfaces X^A and X^B. Then the normal plane which lie unit spacelike normal vectors N^A and

N^B, is a spacelike plane. On the other hand since $\langle N^A, N^B \rangle = \lambda \in (-1,1)$, there exist a $\theta \in (0, \pi)$ such that

$$\langle N^A, N^B \rangle = cos\theta.$$

Then from Eq.(20) and Eq.(22) we have following results:

Corollary 4.1. Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be the two timelike parametric surfaces in Lorentz-Minkowski 3-space L^3 and N^A and N^B are the spacelike unit normal vectors to timelike surfaces X^A and X^B, respectively. Let $\alpha = \alpha(s)$ be the intersection curve of both surfaces X^A and X^B.

1. The curvature and torsion of timelike intersection curve are

$$\kappa^2 = \frac{|(k_n^A)^2 + (k_n^B)^2 - 2\cos(\theta) k_n^A k_n^B|}{sin^2(\theta)}$$

and

$$\tau = \frac{(p_n^A - \cos(\theta) p_n^B)\langle N^A, b \rangle + (p_n^B - \cos(\theta) p_n^A)\langle N^B, b \rangle}{\kappa sin^2(\theta)}$$

2. If $\theta = (\pi/2)$, then

$$\kappa^2 = (k_n^A)^2 + (k_n^B)^2$$

and

$$\tau = \frac{p_n^A \langle N^A, b \rangle + p_n^B \langle N^B, b \rangle}{\kappa}.$$

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